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Economic order quantity under conditionally permissible delay in payments

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Abstract

Within the economic order quantity (EOQ) framework, the main purpose of this paper is to investigate the retailer's optimal replenishment policy under permissible delay in payments. All previously published articles dealing with optimal order quantity with permissible delay in payments assumed that the supplier only offers the retailer fully permissible delay in payments if the retailer ordered a sufficient quantity. Otherwise, permissible delay in payments would not be permitted. However, in this paper, we want to extend this extreme case by assuming that the supplier would offer the retailer partially permissible delay in payments when the order quantity is smaller than a predetermined quantity. Under this condition, we model the retailer's inventory system as a cost minimization problem to determine the retailer's optimal inventory cycle time and optimal order quantity. Three theorems are established to describe the optimal replenishment policy for the retailer. Some previously published results of other researchers can be deduced as special cases. Finally, numerical examples are given to illustrate all these theorems and to draw managerial insights.

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1. Introduction

The traditional economic order quantity (EOQ) model assumes that the retailer must be paid for the items as soon as the items were received. In practice, the supplier hopes to stimulate his products and so he will offer the retailer a delay period, namely, the trade credit period: Before the end of the trade credit period, the retailer can sell the goods and accumulate revenue and earn interest. On the other hand, a higher

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interest is charged if the payment is not settled by the end of the trade credit period. Therefore, it makes economic sense for the retailer to delay the settlement of the replenishment account up to the last moment of the permissible period allowed by the supplier.

Several papers discussing this topic have appeared in the literatures that investigate inventory problems under varying conditions. Some of the prominent papers are discussed below. Goyal [8] established a single-item inventory model for determining the economic ordering quantity in the case that the supplier offers the retailer the opportunity to delay his payment within a fixed time period. Chung [5] simplified the search of the optimal solution for the problem explored by Goyal [8]. Aggarwal and Jaggi [2] considered the inventory model with an exponential deterioration rate under the condition of permissible delay in payments. Jamal et al. [13] then further generalized the model to allow for shortages. Hwang and Shinn [12] developed the model for determining the retailer's optimal price and lot-size simultaneously when the supplier permits delay in payments for an order of a product whose demand rate is a function of constant price elasticity. Jamal et al. [14] formulated a model where the retailer can pay the wholesaler either at the end of the credit period or later, incurring interest charges on the unpaid balances for the overdue period. They developed a retailer's policy for the optimal cycle and payment times for a retailer in a deteriorating-item inventory scenario, in which a wholesaler allows a specified credit period for payment without penalty. Teng [17] assumed that the selling price is not equal to the purchasing price to modify Goyal's model [8]. The important finding from Teng's study [17] is that it makes economic sense for a well-established retailer to order small lot sizes and so take more frequently the benefits of the permissible delay in payments. Chung and Huang [6] extended Goyal [8] to consider the case that the units are replenished at a finite rate under permissible delay in payments and developed an efficient solution-finding procedure to determine the retailer's optimal ordering policy. Huang [9] extended one-level trade credit into two-level trade credit to develop the retailer's replenishment model from the viewpoint of the supply chain. He assumed that not only the supplier offers the retailer trade credit but also the retailer offers the trade credit to his/her customer. This viewpoint reflected more real-life situations in the supply chain model. Khouja [15] showed that for many supply chain configurations, complete synchronization would result in some members of the chain being 'losers' in terms of cost. He used the economic delivery and scheduling problem model and analyzed supply chains dealing with single and multiple components in developing his model. Huang and Chung [11] extended Goyal's model [8] to discuss the replenishment and payment policies to minimize the annual total average cost under cash discount and payment delay from the retailer's point of view. They assumed that the supplier could adopt a cash discount policy to attract retailer to pay the full payment of the amount of purchasing at an earlier time as a means to shorten the collection period. Arcelus et al. [3] modeled the retailer's profit-maximizing retail promotion strategy, when confronted with a vendor's trade promotion offer of credit and/or price discount on the purchase of regular or perishable merchandise. Abad and Jaggi [1] formulated models of seller-buyer relationship, they provided procedures for finding the seller's and buyer's best policies under non-cooperative and cooperative relationship respectively. Huang [10] extended Chung and Huang's model [6], in allowing the retailer adopts different payment policy and finding differences between unit purchase and selling price, and developed an efficient solution-finding procedure to determine the retailer's optimal cycle time and optimal order quantity.

All above published papers assumed that the supplier offer the retailer fully permissible delay in payments independent of the order quantity. Recently, Shinn and Hwang [16] determined the retailer's optimal price and order size simultaneously under the condition of order-size-dependent delay in payments. They assumed that the length of the credit period is a function of the retailer's order size, and also the demand rate is a function of the selling price. Chung and Liao [7] dealt with the problem of determining the economic order quantity for exponentially deteriorating items under permissible delay in payments depending on the ordering quantity and developed an efficient solution-finding procedure to determine the retailer's optimal ordering policy. In this regard, Chang [4] extended Chung and Liao [7] by taking into account inflation and finite time horizon. However, all above published papers dealing with economic order quantity in the presence of

permissible delay in payments assumed that the supplier only offers the retailer fully permissible delay in payments if the retailer orders a sufficient quantity. Otherwise, permissible delay in payments would not be permitted. We know that this policy of the supplier to stimulate the demands from the retailer is very practical. But this is just an extreme case. That is, the retailer would obtain 100% permissible delay in payments if the retailer ordered a large enough quantity. Otherwise, 0% permissible delay in payments would happen.

In reality, the supplier can relax this extreme case to offer the retailer partially permissible delay in payments rather than 0% permissible delay in payments when the order quantity is smaller than a predetermined quantity. That is, the retailer must make a partial payment to the supplier when the order is received to enjoy some portion of the trade credit. Then, the retailer must pay off the remaining balances at the end of the permissible delay period. For example, the supplier provides 100% delay payment permitted if the retailer ordered a sufficient quantity, otherwise only $\alpha\%$ ($0 \leq \alpha \leq 100$) delay payment permitted. From the viewpoint of supplier's marketing policy, the supplier can use the fraction of the permissible delay in payments to agilely control the effects of stimulating the demands from the retailer. This viewpoint is a realistic and novel one in this research field, hence, forms the focus of the present study. Therefore, we ignore the effect of deteriorating item; inflation and finite time horizon similar to most previously published articles. Under these conditions, we model the retailer's inventory system as a cost minimization problem to determine the retailer's optimal inventory cycle time and optimal order quantity. Three theorems are established to describe the optimal replenishment policy for the retailer under the more general framework. Some previously published results of other researchers can be viewed as special cases. Finally, numerical examples are given to illustrate all these theorems and to draw managerial insights.

2. Model formulation and the convexity

In this section, the present study develops a retailer's inventory model under conditionally permissible delay in payments. The following notation and assumptions are used throughout this paper.

Notation

D	demand rate per year
A	ordering cost per order
W	quantity at which the fully delay payments permitted per order
c	unit purchasing price
h	unit stock holding cost per year excluding interest charges
I_e	interest earned per \$ per year
I_k	interest charged per \$ in stocks per year
M	the length of the trade credit period, in years
α	the fraction of the delay payments permitted by the supplier per order, $0 \leq \alpha \leq 1$
T	the length of the cycle time, in years
Q	the order quantity
$TRC(T)$	the annual total relevant cost, which is a function of T
T^*	the optimal cycle time of $TRC(T)$
Q^*	the optimal order quantity = DT^*

Assumptions:

- (1) Replenishments are instantaneous.
- (2) Demand rate, D , is known and constant.
- (3) Shortages are not allowed.

- (4) The inventory system involves only one type of inventory.
- (5) Time horizon is infinite.
- (6) If $Q < W$, i.e. $T < W/D$, the partially delayed payment is permitted. Otherwise, fully delayed payment is permitted. Hence, if $Q \geq W$, pay cQ after M time periods from the time the order is filled. Otherwise, as the order is filled, the retailer must make a partial payment, $(1 - \alpha)cDT$, to the supplier. Then the retailer must pay off the remaining balances, αcDT , at the end of the trade credit period. This assumption constitutes the major difference of the proposed model from previous ones.
- (7) During the time period that the account is not settled, generated sales revenue is deposited in an interest-bearing account.
- (8) $I_k \geq I_e$.

The model:

The annual total relevant cost consists of the following elements. There are three cases to occur: (1) $M \geq W/D$; (2) $M < W/D \leq M/(1 - \alpha)$; and (3) $M/(1 - \alpha) < W/D$.

2.1. Case I: Suppose that $M \geq W/D$

- (1) Annual ordering cost = $\frac{A}{T}$.
- (2) Annual stock holding cost(excluding interest charges) = $\frac{DT}{2}$.
- (3) From assumptions (6) and (7), there are three sub-cases in terms of annual opportunity cost of the capital.
 - (i) $M \leq T$.
The annual opportunity cost of capital = $\frac{cI_k D(T-M)^2}{2T} - cI_e \left(\frac{DM^2}{2}\right) / T$.
 - (ii) $W/D \leq T \leq M$.
The annual opportunity cost of capital = $-cI_e \left[\frac{DT^2}{2} + DT(M - T)\right] / T = -cI_e DT(M - \frac{T}{2}) / T$.
 - (iii) $0 < T < W/D$, as shown in Fig. 1.
The annual opportunity cost of capital = $cI_k \left[\frac{(1-\alpha)^2 DT^2}{2}\right] / T - cI_e DT(M - \frac{T}{2}) / T$.

From the above conditions, the annual total relevant cost for the retailer can be expressed as $TRC(T) = \text{ordering cost} + \text{stock-holding cost} + \text{opportunity cost of capital}$.

$$TRC(T) = \begin{cases} TRC_1(T), & \text{if } M \leq T, & \text{(a)} \\ TRC_2(T), & \text{if } \frac{W}{D} \leq T \leq M, & \text{(b)} \\ TRC_3(T), & \text{if } 0 < T < \frac{W}{D}, & \text{(c)} \end{cases} \quad (1)$$

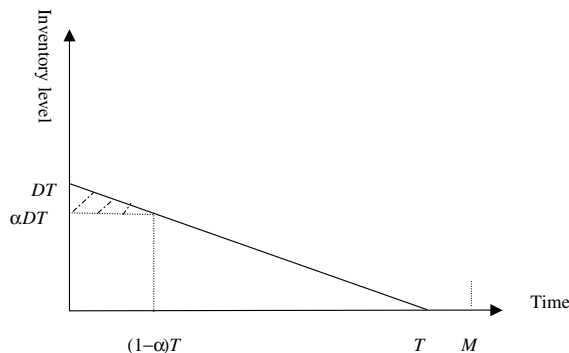


Fig. 1. The inventory level and the total saved amount of interest payable when $0 < T \leq M$.

where

$$TRC_1(T) = \frac{A}{T} + \frac{DTh}{2} + \frac{cI_k D(T - M)^2}{2T} - \frac{DM^2 cI_e}{2T}, \tag{2}$$

$$TRC_2(T) = \frac{A}{T} + \frac{DTh}{2} - cI_e DT \left(M - \frac{T}{2} \right) / T \tag{3}$$

and

$$TRC_3(T) = \frac{A}{T} + \frac{DTh}{2} + (1 - \alpha)^2 cI_k DT^2 / 2T - cI_e DT \left(M - \frac{T}{2} \right) / T. \tag{4}$$

Since $TRC_1(M) = TRC_2(M)$ and $TRC_2(W/D) \leq TRC_3(W/D)$, $TRC(T)$ is continuous except at $T = W/D$. Furthermore, we have $TRC_3(T) \geq TRC_2(T)$ for all $T > 0$ and $TRC_3(T)$ will reduce to $TRC_2(T)$ when $\alpha = 1$. Eqs. (2)–(4) yield

$$TRC'_1(T) = \frac{-[2A + cDM^2(I_k - I_e)]}{2T^2} + \frac{D(h + cI_k)}{2}, \tag{5}$$

$$TRC''_1(T) = \frac{2A + cDM^2(I_k - I_e)}{T^3} > 0, \tag{6}$$

$$TRC'_2(T) = \frac{-A}{T^2} + \frac{D(h + cI_e)}{2}, \tag{7}$$

$$TRC''_2(T) = \frac{2A}{T^3} > 0, \tag{8}$$

$$TRC'_3(T) = -\frac{A}{T^2} + \frac{D\{h + c[(1 - \alpha)^2 I_k + I_e]\}}{2} \tag{9}$$

and

$$TRC''_3(T) = \frac{2A}{T^3} > 0. \tag{10}$$

Eqs. (6), (8) and (10) imply that $TRC_1(T)$, $TRC_2(T)$ and $TRC_3(T)$ are convex on $T > 0$. Moreover, we have $TRC'_1(M) = TRC'_2(M)$ and $TRC'_2(W/D) \neq TRC'_3(W/D)$ except when $\alpha = 1$.

2.2. Case II: Suppose that $M < W/D \leq M/(1 - \alpha)$

If $M < W/D \leq M/(1 - \alpha)$, Eqs. (1(a)–(c)) will be modified as

$$TRC(T) = \begin{cases} TRC_1(T), & \text{if } \frac{W}{D} \leq T, & \text{(a)} \\ TRC_4(T), & \text{if } M \leq T < \frac{W}{D}, & \text{(b)} \\ TRC_3(T), & \text{if } 0 < T \leq M, & \text{(c)} \end{cases} \tag{11}$$

when $M < T < W/D \leq M/(1 - \alpha)$, the annual total relevant cost, $TRC_4(T)$, consists of the following elements:

- (1) Annual ordering cost = $\frac{A}{T}$.
- (2) Annual stock holding cost = $\frac{DTh}{2}$.
- (3) According to assumption (6), the annual opportunity cost of capital (as shown in Fig. 2) = $cI_k \left[\frac{(1-\alpha)^2 DT^2}{2} + \frac{D(T-M)^2}{2} \right] / T - cI_e \left(\frac{DM^2}{2} \right) / T = \frac{cDI_k}{2} [(1 - \alpha)^2 T^2 + (T - M)^2] / T - cI_e \left(\frac{DM^2}{2} \right) / T$.

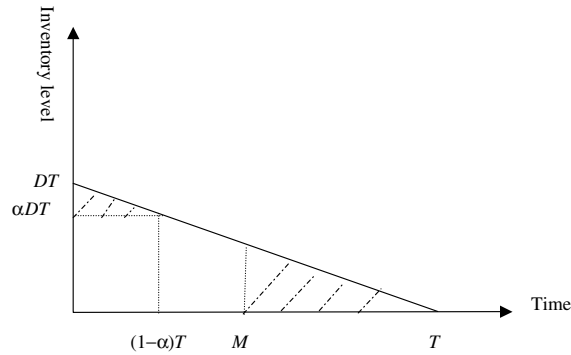


Fig. 2. The inventory level and the total saved amount of interest payable when $M \leq T \leq M/(1 - \alpha)$.

Combining the above conditions, we get

$$TRC_4(T) = \frac{A}{T} + \frac{DT h}{2} + cI_k D[(1 - \alpha)^2 T^2 + (T - M)^2]/2T - cI_e DM^2/2T. \tag{12}$$

Since $TRC_1(W/D) \leq TRC_4(W/D)$ and $TRC_4(M) = TRC_3(M)$, $TRC(T)$ is continuous except at $T = W/D$. Furthermore, we have $TRC_4(T) \geq TRC_1(T)$ for all $T > 0$ and $TRC_4(T)$ will reduce to $TRC_1(T)$ when $\alpha = 1$. Eq. (12) yields

$$TRC'_4(T) = -\left[\frac{2A + cDM^2(I_k - I_e)}{2T^2}\right] + \frac{D\{h + cI_k[1 + (1 - \alpha)^2]\}}{2} \tag{13}$$

and

$$TRC''_4(T) = \frac{2A + cDM^2(I_k - I_e)}{T^3} > 0. \tag{14}$$

Eq. (14) implies that $TRC_4(T)$ is convex on $T > 0$. Moreover, we have $TRC'_1(W/D) \neq TRC'_4(W/D)$ except when $\alpha = 1$ and $TRC'_4(M) = TRC'_3(M)$.

2.3. Case III: Suppose that $M/(1 - \alpha) < W/D$

If $M/(1 - \alpha) < W/D$, Eqs. (1(a)–(c)) and (11(a)–(c)) will be modified as

$$TRC(T) = \begin{cases} TRC_1(T), & \text{if } \frac{W}{D} \leq T, & \text{(a)} \\ TRC_5(T), & \text{if } \frac{M}{1-\alpha} \leq T < \frac{W}{D}, & \text{(b)} \\ TRC_4(T), & \text{if } M \leq T \leq \frac{M}{1-\alpha}, & \text{(c)} \\ TRC_3(T), & \text{if } 0 < T \leq M, & \text{(d)} \end{cases} \tag{15}$$

when $M/(1 - \alpha) \leq T < W/D$, the annual total relevant cost, $TRC_5(T)$, consists of the following elements:

- (1) Annual ordering cost = $\frac{A}{T}$.
- (2) Annual stock holding cost = $\frac{DT h}{2}$.
- (3) According to assumption (6), the annual opportunity cost of capital (as shown in Fig. 3)= $cI_k \left(\frac{DT^2}{2} - \alpha DT M\right) / T - cI_e \left(\frac{DM^2}{2}\right) / T$.

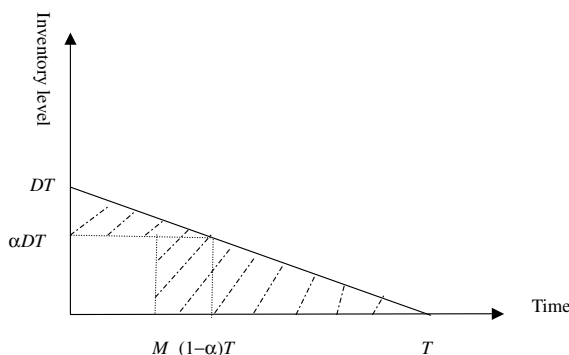


Fig. 3. The inventory level and the total saved amount of interest payable when $M/(1 - \alpha) \leq T$.

Combining the above conditions, we get

$$TRC_5(T) = \frac{A}{T} + \frac{DT h}{2} + cI_k DT \left(\frac{T}{2} - \alpha M \right) / T - cI_e DM^2 / 2T. \tag{16}$$

Since $TRC_1(W/D) \neq TRC_5(W/D)$, $TRC_5(\frac{M}{1-\alpha}) = TRC_4(\frac{M}{1-\alpha})$ and $TRC_4(M) = TRC_3(M)$, $TRC(T)$ is continuous except at $T = W/D$. Furthermore, we have $TRC_5(T) \geq TRC_1(T)$ for $T \geq M/(1 - \alpha)$. Eq. (16) yields

$$TRC'_5(T) = -\left(\frac{2A - cDM^2 I_e}{2T^2} \right) + D \left(\frac{h + cI_k}{2} \right) \tag{17}$$

and

$$TRC''_5(T) = \frac{2A - cDM^2 I_e}{T^3}. \tag{18}$$

Eq. (18) implies that $TRC_5(T)$ is convex on $T > 0$ if $2A - cDM^2 I_e > 0$. Moreover, we have $TRC'_1(W/D) \neq TRC'_5(W/D)$, $TRC'_5(\frac{M}{1-\alpha}) = TRC'_4(\frac{M}{1-\alpha})$ and $TRC'_4(M) = TRC'_3(M)$.

3. Decision rules for the optimal cycle time T^*

In this section, the present study demonstrates the determination of the optimal cycle time for the above three cases, under the condition of minimizing annual total relevant costs.

3.1. Case I: Suppose that $M \geq W/D$

From Eqs. (5), (7) and (9), find T_i^* such that $TRC'_i(T_i^*) = 0$ for each $i = 1, 2, 3$. Then, we can obtain

$$T_1^* = \sqrt{\frac{2A + cDM^2(I_k - I_e)}{D(h + cI_k)}}, \tag{19}$$

$$T_2^* = \sqrt{\frac{2A}{D(h + cI_e)}} \tag{20}$$

and

$$T_3^* = \sqrt{\frac{2A}{D\{h + c[(1 - \alpha)^2 I_k + I_e]\}}}. \quad (21)$$

Eq. (19) gives that the optimal value T^* for the case when $M \leq T$ so that $M \leq T_1^*$. Substituting Eq. (19) into $M \leq T_1^*$, then we can obtain that

$$T_1^* \geq M, \quad \text{if and only if } -2A + DM^2(h + cI_e) \leq 0.$$

Likewise, Eq. (20) gives that the optimal value T^* for the case when $W/D \leq T \leq M$ so that $W/D \leq T_2^* \leq M$. Substituting Eq. (20) into $W/D \leq T_2^* \leq M$, then we can obtain that

$$T_2^* \leq M, \quad \text{if and only if } -2A + DM^2(h + cI_e) \geq 0$$

and

$$T_2^* \geq W/D, \quad \text{if and only if } -2A + \frac{W^2}{D}(h + cI_e) \leq 0.$$

Finally, Eq. (21) gives that the optimal value T^* for the case when $T < W/D$ so that $T_3^* < W/D$. Substituting Eq. (21) into $T_3^* < W/D$, then we can obtain that

$$T_3^* < W/D, \quad \text{if and only if } -2A + \frac{W^2}{D}\{h + c[(1 - \alpha)^2 I_k + I_e]\} > 0.$$

Furthermore, to simplify, we let

$$\Delta_1 = -2A + DM^2(h + cI_e), \quad (22)$$

$$\Delta_2 = -2A + \frac{W^2}{D}(h + cI_e) \quad (23)$$

and

$$\Delta_3 = -2A + \frac{W^2}{D}\{h + c[(1 - \alpha)^2 I_k + I_e]\}. \quad (24)$$

Eqs. (22)–(24) imply that $\Delta_1 \geq \Delta_2$ and $\Delta_3 \geq \Delta_2$. In addition, we know $\text{TRC}_3(T) \geq \text{TRC}_2(T)$ for all $T > 0$ from Eqs. (3) and (4). From above arguments, we can summarize the above results in [Theorem 1](#).

Theorem 1. Suppose that $M \geq W/D$, then

- (A) If $\Delta_1 > 0$, $\Delta_2 > 0$ and $\Delta_3 > 0$, then $\text{TRC}(T^*) = \text{TRC}_3(T_3^*)$ and $T^* = T_3^*$.
- (B) If $\Delta_1 > 0$, $\Delta_2 \leq 0$ and $\Delta_3 > 0$, then $\text{TRC}(T^*) = \text{TRC}_2(T_2^*)$ and $T^* = T_2^*$.
- (C) If $\Delta_1 > 0$, $\Delta_2 \leq 0$ and $\Delta_3 \leq 0$, then $\text{TRC}(T^*) = \text{TRC}_2(T_2^*)$ and $T^* = T_2^*$.
- (D) If $\Delta_1 \leq 0$, $\Delta_2 \leq 0$ and $\Delta_3 > 0$, then $\text{TRC}(T^*) = \min\{\text{TRC}_1(T_1^*), \text{TRC}_3(T_3^*)\}$. Hence, T^* is T_1^* or T_3^* whichever has the least cost.
- (E) If $\Delta_1 \leq 0$, $\Delta_2 \leq 0$ and $\Delta_3 \leq 0$, then $\text{TRC}(T^*) = \text{TRC}_1(T_1^*)$ and $T^* = T_1^*$.

3.2. Case II: Suppose that $M < W/D \leq M/(1 - \alpha)$

If $M < W/D \leq M/(1 - \alpha)$, from Eqs. (11(a)–(c)), we know that

$$\text{TRC}(T) = \begin{cases} \text{TRC}_1(T), & \text{if } \frac{W}{D} \leq T, \\ \text{TRC}_4(T), & \text{if } M \leq T < \frac{W}{D}, \\ \text{TRC}_3(T), & \text{if } 0 < T \leq M. \end{cases}$$

From Eq. (13), find T_4^* such that $\text{TRC}'_4(T_4^*) = 0$. Then, we can obtain

$$T_4^* = \sqrt{\frac{2A + cDM^2(I_k - I_e)}{D\{h + cI_k[1 + (1 - \alpha)^2]\}}}. \tag{25}$$

In a similar fashion, we can obtain following results:

$$T_1^* \geq W/D, \quad \text{if and only if } -[2A + cDM^2(I_k - I_e)] + \frac{W^2}{D}(h + cI_k) \leq 0,$$

$$T_4^* < W/D, \quad \text{if and only if } -[2A + cDM^2(I_k - I_e)] + \frac{W^2}{D}\{h + cI_k[1 + (1 - \alpha)^2]\} > 0$$

and

$$T_4^* \geq M, \quad \text{if and only if } -2A + DM^2\{h + c[(1 - \alpha)^2I_k + I_e]\} \leq 0,$$

$$T_3^* \leq M, \quad \text{if and only if } -2A + DM^2\{h + c[(1 - \alpha)^2I_k + I_e]\} \geq 0.$$

Furthermore, to simplify, we let

$$\Delta_4 = -[2A + cDM^2(I_k - I_e)] + \frac{W^2}{D}(h + cI_k), \tag{26}$$

$$\Delta_5 = -[2A + cDM^2(I_k - I_e)] + \frac{W^2}{D}\{h + cI_k[1 + (1 - \alpha)^2]\} \tag{27}$$

and

$$\Delta_6 = -2A + DM^2\{h + c[(1 - \alpha)^2I_k + I_e]\}. \tag{28}$$

Eqs. (26)–(28) imply that $\Delta_5 \geq \Delta_4$ and $\Delta_5 > \Delta_6$. In addition, we know $\text{TRC}_4(T) \geq \text{TRC}_1(T)$ for all $T > 0$ from Eqs. (2) and (12). From above arguments, we can summarize the above results in **Theorem 2**.

Theorem 2. Suppose that $M < W/D \leq M/(1 - \alpha)$, then

- (A) If $\Delta_4 > 0$, $\Delta_5 > 0$ and $\Delta_6 \geq 0$, then $\text{TRC}(T^*) = \text{TRC}_3(T_3^*)$ and $T^* = T_3^*$.
- (B) If $\Delta_4 > 0$, $\Delta_5 > 0$ and $\Delta_6 < 0$, then $\text{TRC}(T^*) = \text{TRC}_4(T_4^*)$ and $T^* = T_4^*$.
- (C) If $\Delta_4 \leq 0$, $\Delta_5 > 0$ and $\Delta_6 \geq 0$, then $\text{TRC}(T^*) = \min\{\text{TRC}_1(T_1^*), \text{TRC}_3(T_3^*)\}$. Hence, T^* is T_1^* or T_3^* whichever has the least cost.
- (D) If $\Delta_4 \leq 0$, $\Delta_5 > 0$ and $\Delta_6 < 0$, then $\text{TRC}(T^*) = \text{TRC}_1(T_1^*)$ and $T^* = T_1^*$.
- (E) If $\Delta_4 \leq 0$, $\Delta_5 \leq 0$ and $\Delta_6 < 0$, then $\text{TRC}(T^*) = \text{TRC}_1(T_1^*)$ and $T^* = T_1^*$.

3.3. Case III: Suppose that $M/(1 - \alpha) < W/D$

If $M/(1 - \alpha) < W/D$, from Eqs. (15(a)–(d)), we know that

$$\text{TRC}(T) = \begin{cases} \text{TRC}_1(T), & \text{if } \frac{W}{D} \leq T, \\ \text{TRC}_5(T), & \text{if } \frac{M}{1-\alpha} \leq T < \frac{W}{D}, \\ \text{TRC}_4(T), & \text{if } M \leq T \leq \frac{M}{1-\alpha}, \\ \text{TRC}_3(T), & \text{if } 0 < T \leq M. \end{cases}$$

From Eq. (17), find T_5^* such that $\text{TRC}'_5(T_5^*) = 0$. Then, we can obtain

$$T_5^* = \sqrt{\frac{2A - cDM^2I_e}{D(h + cI_k)}}, \quad \text{if } 2A - cDM^2I_e > 0. \tag{29}$$

In a similar fashion, we can obtain following results:

$$T_1^* \geq W/D, \quad \text{if and only if } -[2A + cDM^2(I_k - I_e)] + \frac{W^2}{D}(h + cI_k) \leq 0.$$

$$T_5^* < W/D, \quad \text{if and only if } -[2A - cDM^2I_e] + \frac{W^2}{D}(h + cI_k) > 0$$

and

$$T_5^* \geq M/(1 - \alpha), \quad \text{if and only if } -[2A - cDM^2I_e] + D\left(\frac{M}{1 - \alpha}\right)^2(h + cI_k) \leq 0.$$

$$T_4^* \leq M/(1 - \alpha), \quad \text{if and only if } -[2A - cDM^2I_e] + D\left(\frac{M}{1 - \alpha}\right)^2(h + cI_k) \geq 0$$

and

$$T_4^* \geq M, \quad \text{if and only if } -2A + DM^2\{h + c[(1 - \alpha)^2I_k + I_e]\} \leq 0,$$

$$T_3^* \leq M, \quad \text{if and only if } -2A + DM^2\{h + c[(1 - \alpha)^2I_k + I_e]\} \geq 0.$$

Furthermore, to simplify, we let

$$\Delta_7 = -[2A - cDM^2I_e] + \frac{W^2}{D}(h + cI_k) \tag{30}$$

and

$$\Delta_8 = -[2A - cDM^2I_e] + D\left(\frac{M}{1 - \alpha}\right)^2(h + cI_k). \tag{31}$$

Eqs. (30), (31), (26) and (28) imply that $\Delta_7 > \Delta_8 \geq \Delta_6$ and $\Delta_7 > \Delta_4$. In addition, we know $\text{TRC}_4(T) \geq \text{TRC}_1(T)$ for all $T > 0$ from Eqs. (2) and (12) and $\text{TRC}_5(T) \geq \text{TRC}_1(T)$ for $T \geq M/(1 - \alpha)$ from Eqs. (2) and (16). From above arguments, we can summarize the above results in Theorem 3.

Theorem 3. Suppose that $M/(1 - \alpha) < W/D$, then

- (A) If $\Delta_4 > 0$, $\Delta_7 > 0$, $\Delta_8 \geq 0$ and $\Delta_6 \geq 0$, then $\text{TRC}(T^*) = \text{TRC}_3(T_3^*)$ and $T^* = T_3^*$.
- (B) If $\Delta_4 > 0$, $\Delta_7 > 0$, $\Delta_8 \geq 0$ and $\Delta_6 < 0$, then $\text{TRC}(T^*) = \text{TRC}_4(T_4^*)$ and $T^* = T_4^*$.
- (C) If $\Delta_4 > 0$, $\Delta_7 > 0$, $\Delta_8 < 0$ and $\Delta_6 < 0$, then $\text{TRC}(T^*) = \text{TRC}_5(T_5^*)$ and $T^* = T_5^*$.

- (D) If $\Delta_4 \leq 0, \Delta_7 > 0, \Delta_8 \geq 0$ and $\Delta_6 \geq 0$, then $\text{TRC}(T^*) = \min\{\text{TRC}_1(T_1^*), \text{TRC}_3(T_3^*)\}$. Hence, T^* is T_1^* or T_3^* whichever has the least cost.
- (E) If $\Delta_4 \leq 0, \Delta_7 > 0, \Delta_8 \geq 0$ and $\Delta_6 < 0$, then $\text{TRC}(T^*) = \text{TRC}_1(T_1^*)$ and $T^* = T_1^*$.
- (F) If $\Delta_4 \leq 0, \Delta_7 > 0, \Delta_8 < 0$ and $\Delta_6 < 0$, then $\text{TRC}(T^*) = \text{TRC}_1(T_1^*)$ and $T^* = T_1^*$.
- (G) If $\Delta_4 < 0, \Delta_7 \leq 0, \Delta_8 < 0$ and $\Delta_6 < 0$, then $\text{TRC}(T^*) = \text{TRC}_1(T_1^*)$ and $T^* = T_1^*$.

4. A special case

The value $\alpha = 1$ means that the supplier offers the retailer fully permissible delay in payments. The value $W = 0$ means that the supplier offers the retailer permissible delay in payments independent of the order quantity. Therefore, when $\alpha = 1$ and $W = 0$, Eqs. (1(a)–(c)) will reduced to

$$\text{TRC}(T) = \begin{cases} \text{TRC}_1(T), & \text{if } M \leq T, & \text{(a)} \\ \text{TRC}_2(T), & \text{if } 0 < T \leq M, & \text{(b)} \end{cases} \tag{32}$$

Eqs. (32(a) and (b)) will be consistent with Eqs. (1) and (4) in Goyal [8], respectively. Hence, Goyal [8] will be a special case of this paper. From Eq. (22), we know $\Delta_1 = -2A + DM^2(h + cI_e)$. If we let $\Delta = -2A + DM^2(h + cI_e)$, Theorem 1 can be modified as follows:

Theorem 4

- (A) If $\Delta > 0$, then $T^* = T_2^*$.
- (B) If $\Delta < 0$, then $T^* = T_1^*$.
- (C) If $\Delta = 0$, then $T^* = T_2^* = T_1^* = M$.

Theorem 4 has been discussed in Chung [5]. Hence, Theorem 1 in Chung [5] is a special case of Theorem 1 of the present study.

5. Numerical examples

In this section, the present study provides the following numerical examples to illustrate all the theoretical results as reported in Section 3. For convenience, the values of the parameters are selected randomly. The optimal cycle time and optimal order quantity for different parameters of α (0.2, 0.5, 0.8), W (100, 200, 300) and c (10, 30, 50) are shown in Table 1. The following inferences can be made based on Table 1.

- (1) For fixed W and c , increasing the value of α will result in a significant increase in the value of the optimal order quantity and a significant decrease in the value of the annual total relevant costs as the retailer’s order quantity is smaller and only the partially delayed payment is permitted. For example, when $W = 300, c = 50$ and α increases from 0.2 to 0.5, the optimal order quantity will increase 9.51% $((101.3 - 92.5)/92.5)$ and the annual total relevant costs will decrease 14.24% $((661.67 - 567.42)/661.67)$. However, if the fully delayed payment is permitted, the optimal order quantity and the annual total relevant cost are independent of the value of α . It implies that the retailer will order a larger quantity since the retailer can enjoy greater benefits when the fraction of the delay payments permitted is increasing. So the supplier can use the policy of increasing α to stimulate the demands from the retailer. Consequently, the supplier’s marketing policy under partially permissible delay in payments will be more agile than fully permissible delay in payments.
- (2) For fixed α and c , increasing the value of W will result in a significant decrease in the value of the optimal order quantity and a significant increase in the value of the annual total relevant costs. For example, when $\alpha = 0.2, c = 50$ and W increases from 100 to 200, the optimal order quantity will

Table 1
Optimal solutions under different parametric values

α	W	c	W/D	$M/(1 - \alpha)$	Judgments of $A_i (i = 1-8)$	T^*	Q^*	TRC(T^*)	Theorem
0.2	100	10	0.1	0.15	$A_1 < 0, A_2 < 0, A_3 < 0$	$T_1^* = 0.13186$	131.9	671.15	1-(F)
		30	0.1	0.15	$A_1 > 0, A_2 < 0, A_3 < 0$	$T_2^* = 0.11868$	118.7	590.62	1-(C)
		50	0.1	0.15	$A_1 > 0, A_2 < 0, A_3 > 0$	$T_2^* = 0.10847$	108.5	501.95	1-(B)
	200	10	0.2	0.15	$A_4 > 0, A_7 > 0, A_8 > 0, A_6 < 0$	$T_4^* = 0.12534$	125.3	712.28	3-(B)
		30	0.2	0.15	$A_4 > 0, A_7 > 0, A_8 > 0, A_6 > 0$	$T_3^* = 0.10529$	105.3	697.74	3-(A)
		50	0.2	0.15	$A_4 > 0, A_7 > 0, A_8 > 0, A_6 > 0$	$T_3^* = 0.09245$	92.5	661.67	3-(A)
	300	10	0.3	0.15	$A_4 > 0, A_7 > 0, A_8 > 0, A_6 < 0$	$T_4^* = 0.12534$	125.3	712.28	3-(B)
		30	0.3	0.15	$A_4 > 0, A_7 > 0, A_8 > 0, A_6 > 0$	$T_3^* = 0.10529$	105.3	697.74	3-(A)
		50	0.3	0.15	$A_4 > 0, A_7 > 0, A_8 > 0, A_6 > 0$	$T_3^* = 0.09245$	92.5	661.67	3-(A)
0.5	100	10	0.1	0.24	$A_1 < 0, A_2 < 0, A_3 < 0$	$T_1^* = 0.13186$	131.9	671.15	1-(F)
		30	0.1	0.24	$A_1 > 0, A_2 < 0, A_3 < 0$	$T_2^* = 0.11868$	118.7	590.62	1-(C)
		50	0.1	0.24	$A_1 > 0, A_2 < 0, A_3 < 0$	$T_2^* = 0.10847$	108.5	501.95	1-(C)
	200	10	0.2	0.24	$A_4 > 0, A_5 > 0, A_6 < 0$	$T_4^* = 0.12919$	129.2	687.47	2-(B)
		30	0.2	0.24	$A_4 > 0, A_5 > 0, A_6 > 0$	$T_3^* = 0.11287$	112.9	634.00	2-(A)
		50	0.2	0.24	$A_4 > 0, A_5 > 0, A_6 > 0$	$T_3^* = 0.10127$	101.3	567.42	2-(A)
	300	10	0.3	0.24	$A_4 > 0, A_7 > 0, A_8 > 0, A_6 < 0$	$T_4^* = 0.12919$	129.2	687.47	3-(B)
		30	0.3	0.24	$A_4 > 0, A_7 > 0, A_8 > 0, A_6 > 0$	$T_3^* = 0.11287$	112.9	634.00	3-(A)
		50	0.3	0.24	$A_4 > 0, A_7 > 0, A_8 > 0, A_6 > 0$	$T_3^* = 0.10127$	101.3	567.42	3-(A)
0.8	100	10	0.1	0.6	$A_1 < 0, A_2 < 0, A_3 < 0$	$T_1^* = 0.13186$	131.9	671.15	1-(F)
		30	0.1	0.6	$A_1 > 0, A_2 < 0, A_3 < 0$	$T_2^* = 0.11868$	118.7	590.62	1-(C)
		50	0.1	0.6	$A_1 > 0, A_2 < 0, A_3 < 0$	$T_2^* = 0.10847$	108.5	501.95	1-(C)
	200	10	0.2	0.6	$A_4 > 0, A_5 > 0, A_6 < 0$	$T_4^* = 0.13142$	131.4	673.78	2-(B)
		30	0.2	0.6	$A_4 > 0, A_5 > 0, A_6 > 0$	$T_3^* = 0.11769$	117.7	597.71	2-(A)
		50	0.2	0.6	$A_4 > 0, A_5 > 0, A_6 > 0$	$T_3^* = 0.10721$	107.2	512.74	2-(A)
	300	10	0.3	0.6	$A_4 > 0, A_5 > 0, A_6 < 0$	$T_4^* = 0.13142$	131.4	673.78	2-(B)
		30	0.3	0.6	$A_4 > 0, A_5 > 0, A_6 > 0$	$T_3^* = 0.11769$	117.7	597.71	2-(A)
		50	0.3	0.6	$A_4 > 0, A_5 > 0, A_6 > 0$	$T_3^* = 0.10721$	107.2	512.74	2-(A)

Let $A = \$50/\text{order}$, $D = 1000 \text{ units/year}$, $h = \$5/\text{unit/year}$, $I_k = \$0.1/\text{\$/year}$, $I_e = \$0.07/\text{\$/year}$ and $M = 0.12 \text{ year}$.

decrease 14.75% $((108.5 - 92.5)/108.5)$ and the annual total relevant costs will increase 31.82% $((661.67 - 501.95)/501.95)$. It implies that the retailer will not order a quantity as large as the minimum order quantity as required to obtain fully permissible delay in payments. Hence, the effect of stimulating the demands from the retailer turns negative when the supplier adopts a policy to increase the value of W .

- (3) Last, for fixed α and W , increasing the value of c will result in a significant decrease in the value of the optimal order quantity and a significant decrease in the value of the annual total relevant cost. For example, when $\alpha = 0.2$, $W = 100$ and c increases from 10 to 50, the optimal order quantity will decrease 17.74% $((131.9 - 108.5)/131.9)$ and the annual total relevant costs will decrease 25.21% $((671.15 - 501.95)/671.15)$. This result implies that the retailer will order a smaller quantity to enjoy the benefits of either the fully or partially the permissible delay in payments more frequently in the presence of an increased unit purchasing price.

6. Summary and conclusions

The supplier offers the permissible delay in payments to the retailer in order to stimulate the demand. Hence, the assumption in previously published results that the fully permissible delay in payments is permitted under a sufficient quantity is practical. On the other hand, the permissible delay in payments will not be permitted when the order quantity is smaller than a predetermined quantity obviously is an extreme case.

In this paper, the proposed model allows the supplier to offer an alternative policy, i.e., partially permissible delay in payments, when the retailer's order quantity is not large enough to get the fully permissible delay in payments. Viewed from such perspective, we model the inventory system to take care of the following states: The retailer under fully permissible delay in payments if the retailer orders a large quantity; Otherwise, the retailer will just obtain partially permissible delay in payments. In addition, we establish three effective and easy-to-use theorems to help the retailer to find the optimal replenishment policy. Finally, some numerical examples are provided to illustrate all the theorems, and to obtain the following managerial insights: (1) a higher value of the fraction of the delay payments permitted brings about a larger order quantity and smaller annual total relevant costs; (2) a higher value of the minimum order quantity as required to obtain fully permissible delay in payments brings about a smaller order quantity and larger annual total relevant costs; (3) a higher value of unit purchasing price brings about a smaller order quantity and smaller annual total relevant costs.

From the viewpoint of supplier's marketing policy, the supplier can use the fraction of the delay payments permitted to control more agilely the effects of stimulating the demand from the retailer. For example, the supplier can offer the larger fraction of the delay payments permitted to stimulate the larger order quantity from the retailer. On the other hand, the supplier can use the smaller fraction of the delay payments permitted to decrease the order quantity of the retailer. As such, the more realistic and flexible marketing policy is valuable to the supplier.

The proposed model can be extended in several ways. For instance, we may generalize the model to allow for shortages, deteriorating item, probabilistic demand, time value of money, finite time horizon, and finite replenishment rate worthy of future research.

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